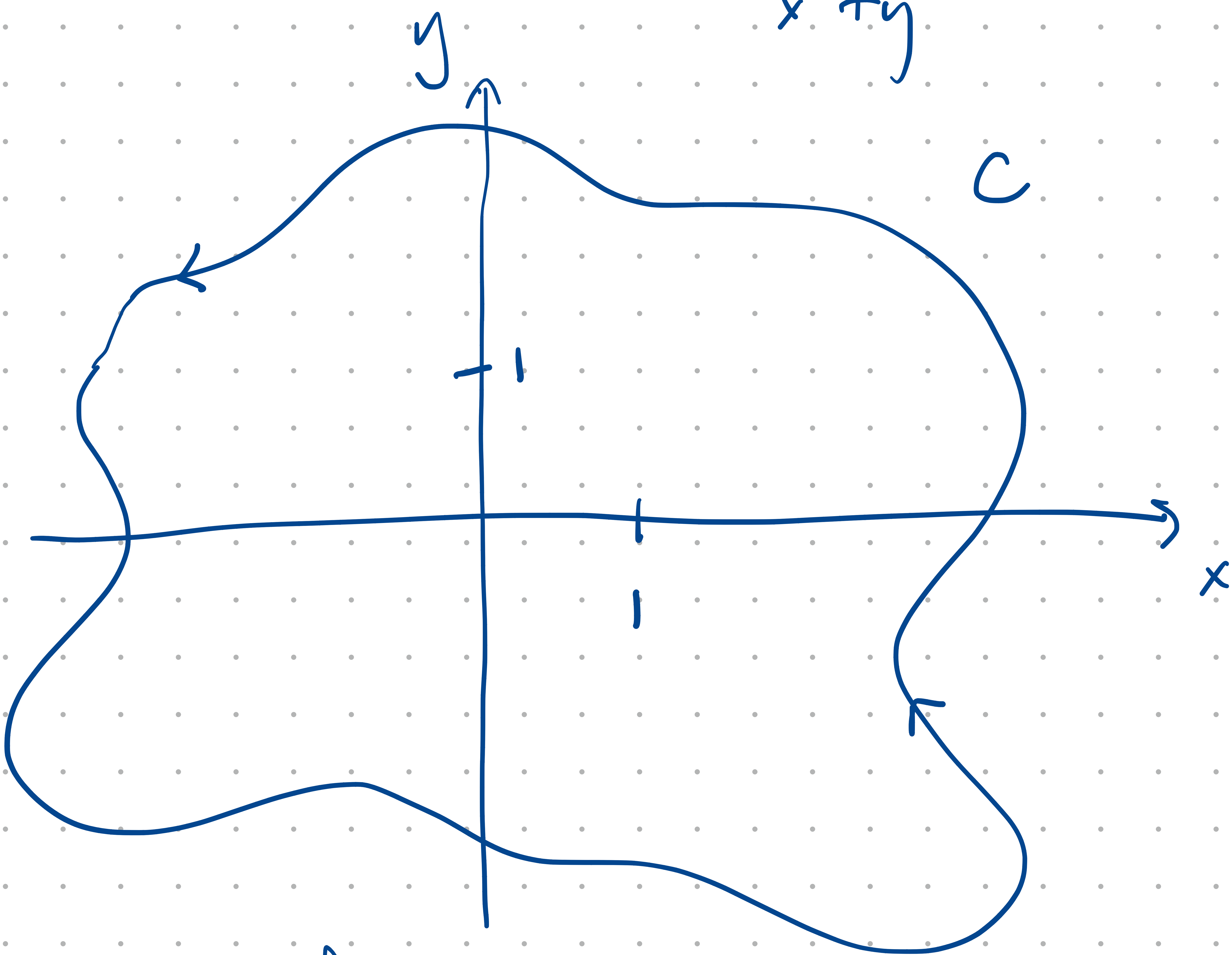
Stewart p. 1150 #38

Let $\vec{F} = \dfrac{(2x^3+2xy^2-2y)\hat{\imath} + (2y^3+2x^2y+2x)\hat{\jmath}}{x^2+y^2}$

ie. $= \langle P, Q \rangle$ where $P = \dfrac{2x^3+2xy^2-2y}{x^2+y^2}$

$Q = \dfrac{2y^3+2x^2y+2x}{x^2+y^2}$.



What is $\oint_C \vec{F}\cdot d\vec{r}$?

First thought: this problem looks impossible, b/c we don't know what exactly the curve C is.

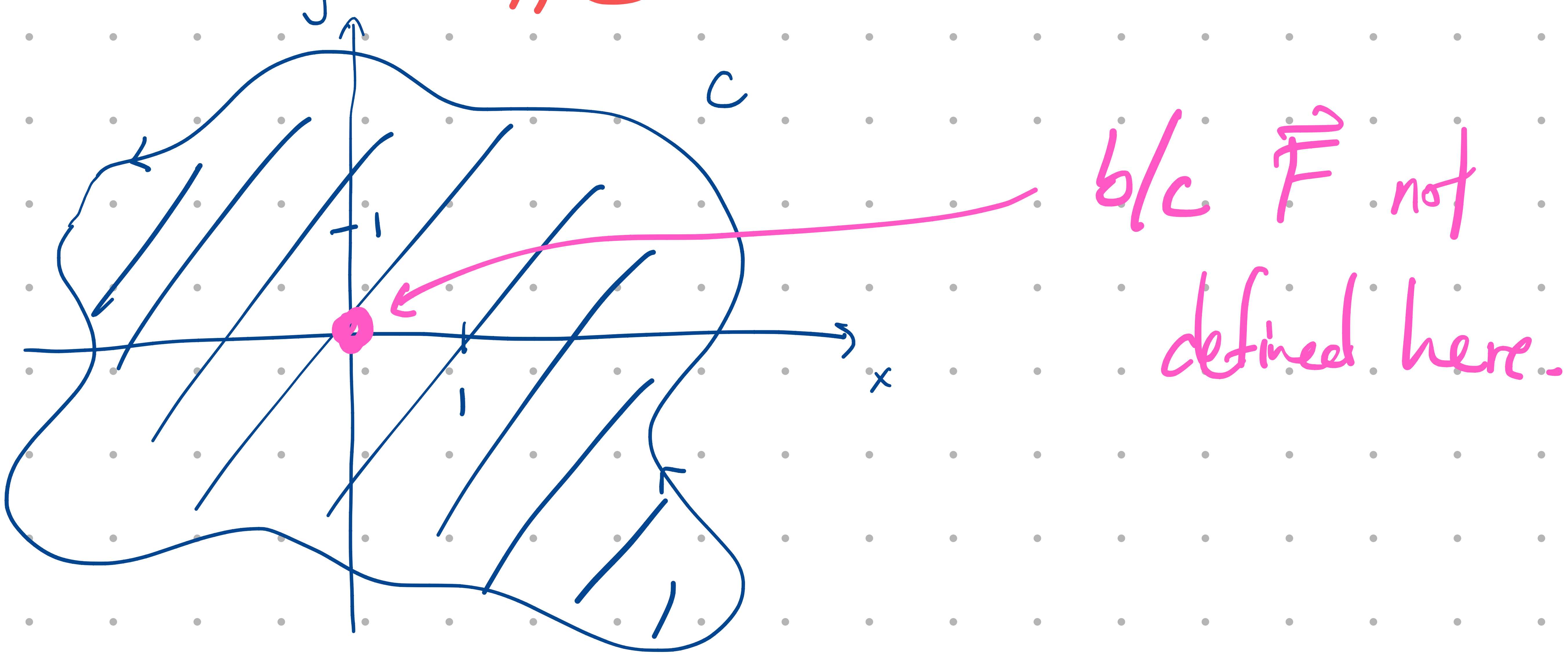
Second thought: Maybe C doesn't matter b/c $\vec{F}$ is conservative (path-independent) so $\oint_C \vec{F} \cdot d\vec{r} = 0$.

So along this train of thought we compute $Q_x - P_y$

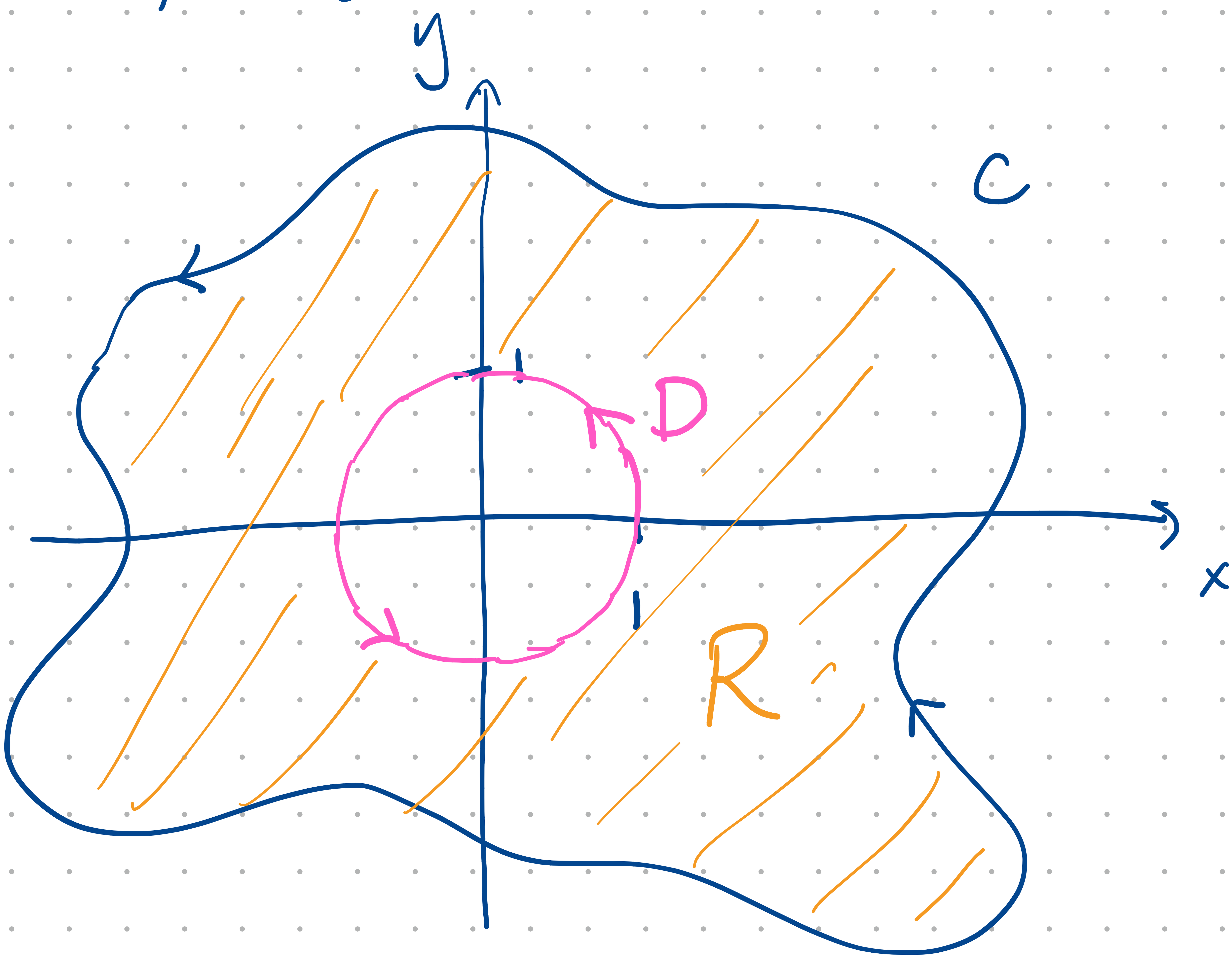
Turns out $Q_x - P_y = 0$ (check!!)

⚠ However, we cannot conclude $\vec{F}$ is conservative since $\text{domain}(\vec{F}) = \mathbb{R}^2 - (0,0)$ is not simply conn.

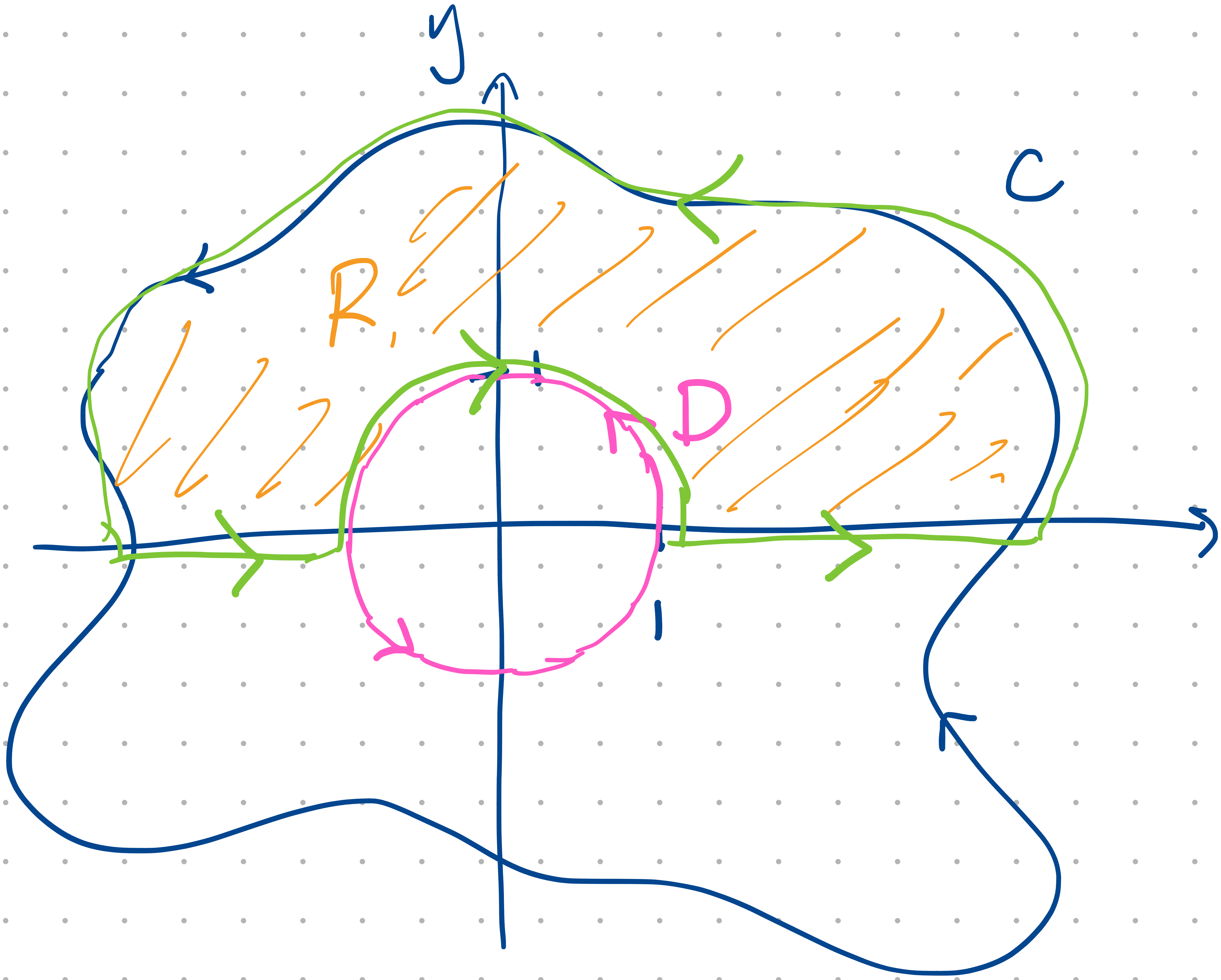
Similarly we can't apply Green's Thm to

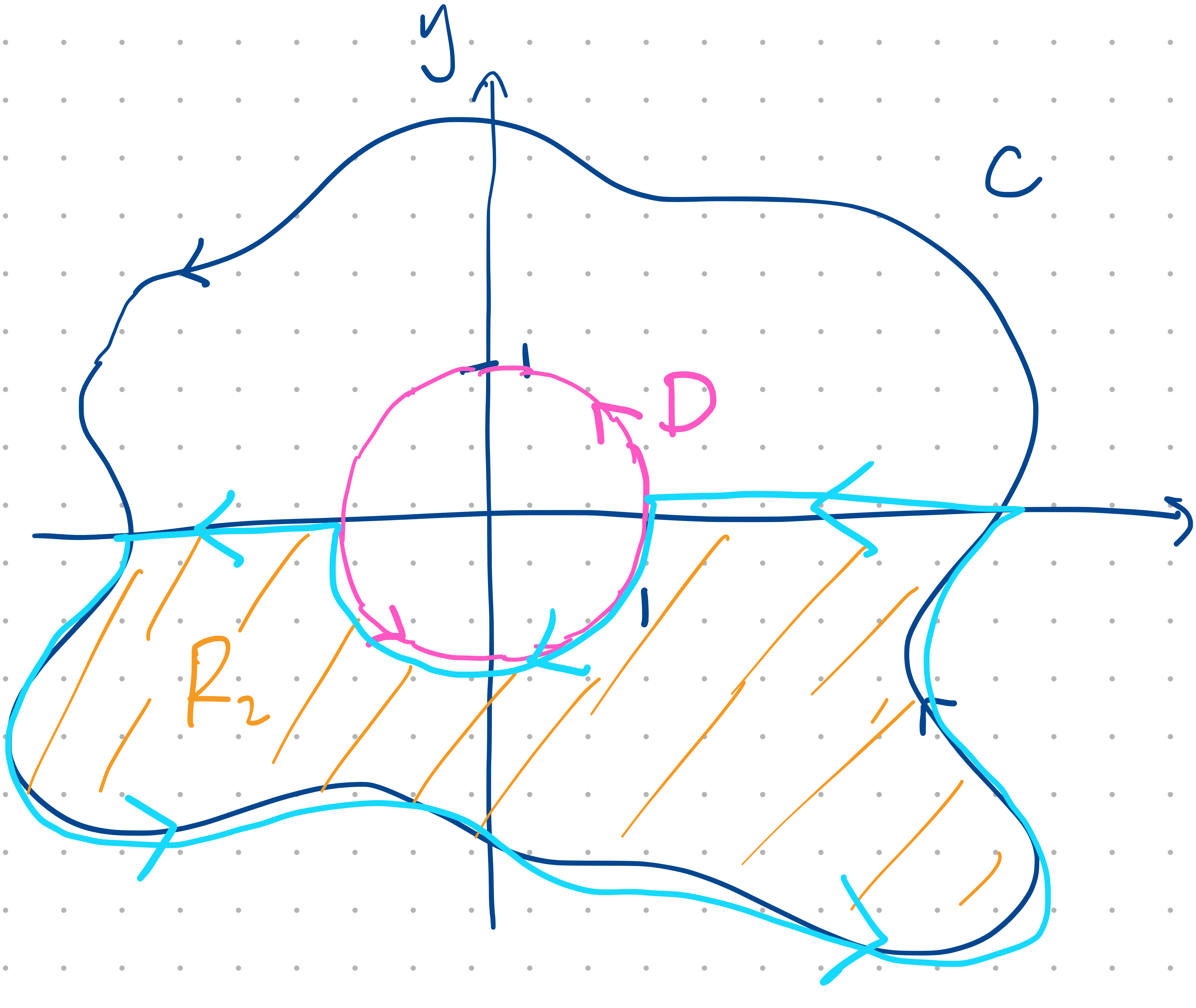


(For the following, refer to Example 5 in 16.4 and the discussion preceding it.)



Key idea: Apply Green's Thm. to R instead.


y
C
R_1
D
1
1
x


y
C
D
1
1
R_2
x

Claim: Boundary of R = C together with

(positively oriented)

Notation: ∂R

$-D$ (reverse of D)

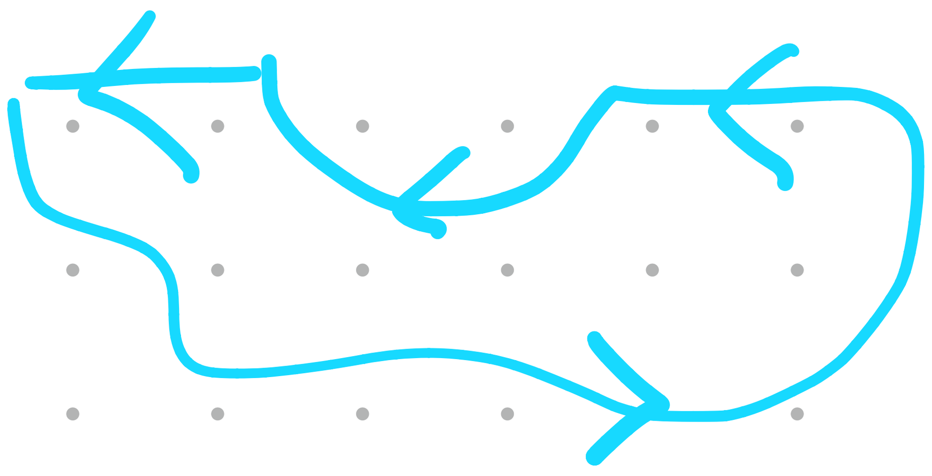
$-D$: unit circle CW

D: unit circle CCW

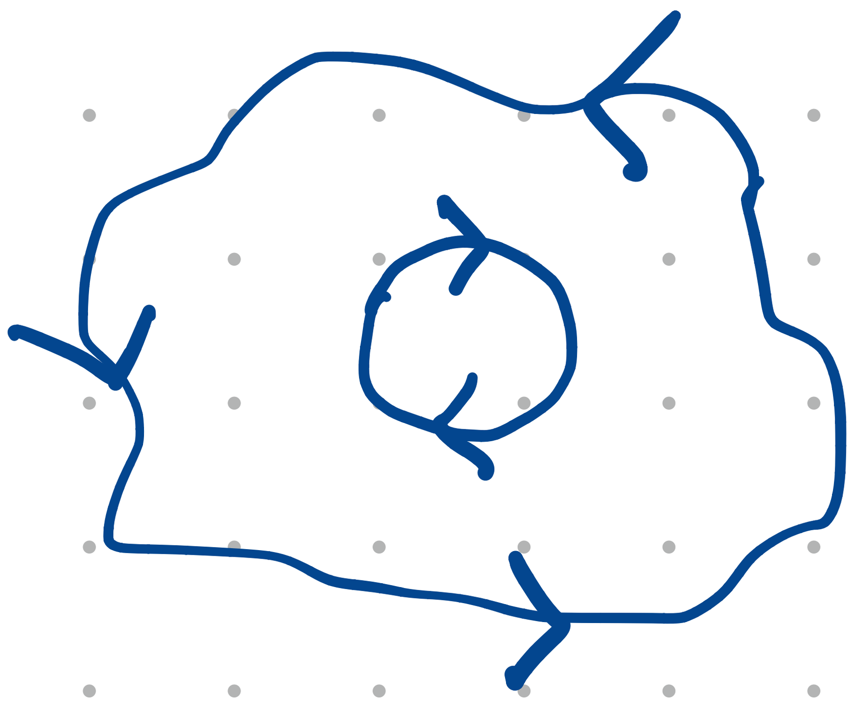
Explanation:

$\partial R_1 =$

$\partial R_2 =$



together: $\partial R =$



$$\oint_C \vec{F}\cdot d\vec{r} + \underbrace{\oint \vec{F}\cdot dr}_{-D} \overset{\text{Green's Thm}}{=} \iint_R (Q_x - P_y)\,dx\,dy$$

$$= -\oint_D \vec{F}\cdot d\vec{r} \qquad \underset{Q_x - P_y = 0 \text{ in this problem}}{=} 0$$

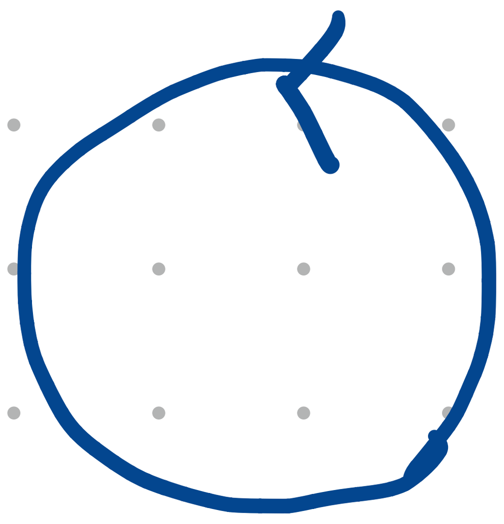
Rearrange:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_D \vec{F} \cdot d\vec{r}$$

actually know this curve!

Now want integral of

$$\vec{F} = \left\langle \frac{2x^3+2xy^2-2y}{x^2+y^2}, \frac{2y^3+2x^2y+2x}{x^2+y^2} \right\rangle$$

on D:  $x^2+y^2=1$.

$$\oint_D \left\langle \frac{2x^3+2xy^2-2y}{x^2+y^2}, \frac{2y^3+2x^2y+2x}{x^2+y^2} \right\rangle \cdot d\vec{r}$$

$x^2+y^2=1$

$$= \oint_D \left\langle 2x^3+2xy^2-2y, 2y^3+2x^2y+2x \right\rangle \cdot d\vec{r}$$

I can actually use Green's thm now!!

$$= \iint_{x^2+y^2\le 1} \left((4xy+2)-(4xy-2)\right) dx\,dy$$

$$= \iint_{x^2+y^2\le 1} 4\, dx\,dy = 4 \text{ Area} = \boxed{4\pi}$$